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Force and impulse from an Aharonov–Bohm flux line

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Abstract

We calculate the force operator for a charged particle in the field of an Aharonov–Bohm flux line. Formally this is the Lorentz force, with the magnetic field operator modified to include quantum corrections due to anomalous commutation relations. For stationary states, the magnitude of the force is proportional to the product of the wavenumber k with the amplitudes of the 'pinioned' components, the two angular momentum components whose azimuthal quantum numbers are closest to the flux parameter α. The direction of the force depends on the relative phase of the pinioned components. For paraxial beams, the transverse component of our expression gives an exact version of Shelankov's formula (Shelankov A 1998 *Europhys. Lett.* **43** 623–8), while the longitudinal component gives the force along the beam.

Nonstationary states are treated by integrating the force operator in time to obtain the impulse operator. Expectation values of the impulse are calculated for two kinds of wavepacket. For slow wavepackets, which spread faster than they move, the impulse is inversely proportional to the distance from the flux line. For fast wavepackets, which spread only negligibly before their closest approach to the flux line, the impulse is proportional to the probability density transverse to the incident direction evaluated at the flux line. In this case, the transverse component of the impulse gives a wavepacket analogue of Shelankov's formula. The direction of the impulse for both kinds of wavepacket is flux dependent.

We give two derivations of the force and impulse operators, the first a simple derivation based on formal arguments, and the second a rigorous calculation of wavepacket expectation values. We also show that the same expressions for the force and impulse are obtained if the flux line is enclosed in an impenetrable cylinder, or distributed uniformly over a flux cylinder, in the limit that the radius of the cylinder goes to zero.

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1. Introduction

There have been a number of investigations of the force exerted on a charged particle by an Aharonov–Bohm flux line. Classically, of course, there is no force, so it, like the Aharonov–

Bohm effect itself, is essentially quantum mechanical, vanishing as $\hbar \to 0$. Olariu and Popescu (1983, 1985) show that for certain localized wavepackets (these are the fast wavepackets described in section 4.2 below), the force, along with the momentum it imparts, is negligible unless the centre of the wavepacket hits the flux line. Nielson and Hedegård (1995) and Shelankov (2000) compute matrix elements of the force operator for stationary states of the same energy. Shelankov (1998) calculates the transverse force on a stationary incident beam of finite angular width using a paraxial approximation, a result we refer to as *Shelankov's formula*. This use of the paraxial approximation has been justified by Berry (1999), who computes the asymptotic deflection of the beam from an exact representation. Peshkin (1981, 1989) computes the expectation value of the force when the flux line is enclosed in an impenetrable cylinder. Recent interest in this problem has been stimulated by the analogy with the Iordanskii force (Iordanskii 1966) exerted on phonons by a vortex in a superfluid (see, e.g., Sonin 1976, 1997, Thouless *et al* 1997, Stone 2000), which has been the subject of some recent debate.

In this paper we add to these investigations in several ways. First, we obtain exact expressions for the Lorentz force operator due to an Aharonov–Bohm flux line and its matrix elements between arbitrary stationary states; the restriction to on-shell matrix elements, between stationary states of the same energy, agrees with previous results. We also obtain an exact expression for the time integral of the force, the impulse operator, and compute its expectation values for various kinds of wavepacket. We show that the force operator can be simply derived, formally at least, from purely kinematic considerations, and give a mathematically rigorous demonstration to justify the results obtained in this way.

The paper is organized as follows. In section 2, we give a formal derivation of the (vector) Lorentz force operator due to an Aharonov–Bohm flux line. Pointing out that the nominal magnetic field operator, $(\alpha hc/e)\delta^2(\boldsymbol{r})$, is incompatible with gauge invariance, we show that that a modification of the canonical commutation relations restores gauge invariance and yields the Lorentz force. An explicit formula for the expectation value for stationary states follows directly. On-shell matrix elements between stationary states of the same energy are seen to coincide with previous results. The transverse component of the force is shown to coincide with Shelankov's formula in the paraxial limit (see also Shelankov 2000). We then compute (section 3) the impulse operator, the integral of the force operator in infinite forward and backward time, in the position representation. In section 4 we compute the leading-order expectation value of the impulse for two kinds of wavepacket. For slow wavepackets (section 4.1), which spread more quickly than they move, the impulse is inversely proportional to the initial distance between the wavepacket and the flux line, with magnitude and direction depending periodically on the flux parameter α. By treating the deflection of a slow wavepacket as a classical scattering, we obtain an expression for the scattering cross-section which, surprisingly, coincides with the exact result. Fast wavepackets (section 4.2) move more quickly than they spread, so much so that the fractional increase in their width as they pass the flux line is small. In this case, the impulse is proportional to the transverse probability density at the flux line, and its transverse component gives a wavepacket analogue of Shelankov's formula.

In the appendix, we give a rigorous derivation of the force and impulse expectation values for a class of well behaved wavefunctions. In section 5, we compute matrix elements of the force operator for two standard regularizations of the Aharonov–Bohm flux line, the first where the flux is enclosed in an impenentrable cylinder, and the second where it is distributed uniformly in a cylindrical tube. The Aharonov–Bohm results of sections 2 and 3 are recovered in the limit that the radius of the cylinder goes to zero.

2. Force operator

Consider a particle of charge e and mass M moving in the xy-plane in a magnetic field along $\hat{\boldsymbol{z}}$. Quantum mechanically, the particle is described by the Hamiltonian $H = \frac{1}{2}\boldsymbol{MV}\cdot\boldsymbol{V}$, where $\boldsymbol{MV} = \boldsymbol{P} - e\boldsymbol{A}/c$ is the kinetic momentum and $\boldsymbol{A}(\boldsymbol{r})$ is the vector potential. The force, i.e. the rate of change of the kinetic momentum, is given by the appropriately symmetrized Lorentz force operator,

$$\boldsymbol{F} = \frac{1}{\mathrm{i}\hbar}[\boldsymbol{MV}, H] = \frac{e}{2c}(\boldsymbol{V}\wedge(B\,\hat{\boldsymbol{z}}) - (B\,\hat{\boldsymbol{z}})\wedge\boldsymbol{V}) \tag{1}$$

where the magnetic field operator is defined by

$$B = \frac{\Phi_0}{2\pi\mathrm{i}\hbar^2}[MV_x, MV_y]. \tag{2}$$

Here $\Phi_0 = 2\pi\hbar c/e$ is the magnetic flux quantum. The usual commutation relations for position and canonical momentum lead to the usual relation between the magnetic field and the vector potential, namely $B\,\hat{\boldsymbol{z}} = \boldsymbol{\nabla}\wedge\boldsymbol{A}$.

However, this relation is incorrect for the vector potential of an Aharonov–Bohm flux line (since we are restricting to the plane, we should perhaps say 'flux point', but we will follow conventional usage). For a flux line at the origin of strength $\alpha\Phi_0$, and in a circularly symmetric gauge, the vector potential is given by

$$\boldsymbol{A}(\boldsymbol{r}) = \alpha\frac{\Phi_0}{2\pi r}\,\hat{\boldsymbol{\phi}}. \tag{3}$$

As is well known, physically meaningful quantities depend only on the fractional part, $\tilde{\alpha}$, of the flux parameter, α; a unit shift in the flux parameter, $\alpha \to \alpha + 1$, is equivalent to the gauge transformation $\psi \to U\psi$, where

$$(U\psi)(r,\phi) = \mathrm{e}^{\mathrm{i}\phi}\psi(r,\phi). \tag{4}$$

As a consequence, a physically observable operator $\mathrm{O}(\alpha)$ which depends on the flux parameter must transform under U according to

$$U\mathrm{O}(\alpha)U^{\dagger} = \mathrm{O}(\alpha+1). \tag{5}$$

For example, the kinetic momentum $\boldsymbol{MV}$ satisfies this relation, as is easily verified. As the magnetic field operator (2) is expressed in terms of the commutator of components of kinetic momentum, it must also satisfy (5). However, $\boldsymbol{\nabla}\wedge\boldsymbol{A}$, which is given by $\alpha\Phi_0\delta^2(\boldsymbol{r})\,\hat{\boldsymbol{z}}$, clearly does not satisfy (5); it is invariant under the gauge transformation U (as is any operator which depends only on position), but is not periodic in α. It follows that substituting $\nabla\wedge\boldsymbol{A}$ for the magnetic field in (1) cannot give the correct expression for the Lorentz force operator.

The problem is caused, of course, by the singularity at $r = 0$, and can be avoided by an explicit calculation of the time rate of change of the expectation value of the kinetic momentum for suitably chosen wavefunctions. This is done in the appendix. However, we can obtain the same result more easily and more directly from formal arguments. If the usual canonical commutation relations lead to a magnetic field operator which does not transform correctly under gauge transformations, then the canonical commutation relations must be modified by the flux line. In particular, we show below that, formally, the components of the canonical momentum, p_x and p_y, do not commute—equivalently, the partial derivatives ∂_x and ∂_y do not commute—in the presence of nonzero flux.

First, we note that the partial derivatives ∂_x and ∂_y certainly commute when applied to smooth wavefunctions. That is,

$$[\partial_x, \partial_y]\psi = 0 \tag{6}$$

for $\psi(\boldsymbol{r})$ smooth. Applying the gauge transformation U to this relation m times, we obtain

$$[U^m \partial_x U^{\dagger m}, U^m \partial_y U^{\dagger m}](\mathrm{e}^{\mathrm{i}m\phi}\psi) = 0. \tag{7}$$

The partial derivatives transform according to

$$U^m \partial_x U^{\dagger m} = \partial_x - \mathrm{i}m\partial_x\phi \qquad U^m \partial_y U^{\dagger m} = \partial_y - \mathrm{i}m\partial_y\phi. \tag{8}$$

Substituting (8) into (7), and using the differential version of Stokes' theorem,

$$[\partial_x, \partial_y]\phi = (\boldsymbol{\nabla} \wedge \boldsymbol{\nabla})_z\phi = 2\pi\delta^2(\boldsymbol{r}) \tag{9}$$

we obtain

$$[\partial_x, \partial_y](\mathrm{e}^{\mathrm{i}m\phi}\psi) = 2\pi\delta^2(\boldsymbol{r})\mathrm{i}m\mathrm{e}^{\mathrm{i}m\phi}\psi = 2\pi\delta^2(\boldsymbol{r})\partial_\phi(\mathrm{e}^{\mathrm{i}m\phi}\psi) \tag{10}$$

where the second equality follows because $\partial_\phi\psi$ vanishes at the origin for $\psi(\boldsymbol{r})$ smooth. This implies, formally, the operator relation

$$[\partial_x, \partial_y] = 2\pi\delta^2(\boldsymbol{r})\partial_\phi \tag{11}$$

or, equivalently,

$$[P_x, P_y] = -2\pi\mathrm{i}\hbar\delta^2(\boldsymbol{r})L \tag{12}$$

where $L = (\hbar/\mathrm{i})\partial_\phi$ is the canonical angular momentum.

Using the modified commutation relation (12) to evaluate the magnetic field operator (2), we obtain, instead of the classical relation $B = \alpha\Phi_0\delta^2(\boldsymbol{r})$, the result

$$B = \frac{\Phi_0}{2\pi\mathrm{i}\hbar^2}[P_x, P_y] + (\nabla \wedge \boldsymbol{A})_z = -\frac{\Phi_0}{\hbar}\delta^2(\boldsymbol{r})\Lambda \tag{13}$$

where

$$\Lambda\,\hat{\boldsymbol{z}} = \boldsymbol{r} \wedge M\boldsymbol{V} = (L - \alpha\hbar)\,\hat{\boldsymbol{z}} \tag{14}$$

is the kinetic angular momentum. It is evident that Λ, and hence B, satisfies the transformation law (5).

Throughout this and the following sections, it will be convenient to represent vectors in the xy-plane as complex scalars. For example, $\boldsymbol{W} = W_x\,\hat{\boldsymbol{x}} + W_y\,\hat{\boldsymbol{y}}$ will be represented by $\mathcal{W} = W_x + \mathrm{i}W_y$. If $\boldsymbol{W}$ is a vector of Hermitian operators (as opposed to real scalars), then $\mathcal{W}$ is the non-Hermitian scalar operator whose Hermitian and anti-Hermitian parts are W_x and $\mathrm{i}W_y$ respectively. In this way, the kinetic momentum $M\boldsymbol{V}$ is represented by the scalar operator

$$M\mathcal{V} = \frac{\hbar}{\mathrm{i}}\mathrm{e}^{\mathrm{i}\phi}\left(\partial_r + \frac{\mathrm{i}\partial_\phi + \alpha}{r}\right). \tag{15}$$

It is useful to note that, in general, $\hat{\boldsymbol{z}} \wedge \boldsymbol{W}$ is represented by $\mathrm{i}\mathcal{W}$.

We proceed to compute the expectation value of the Lorentz force. Substituting (13) for the magnetic field and (15) for the kinetic momentum into the expression (1), we obtain

$$\begin{aligned}\langle\psi|\mathcal{F}|\psi\rangle &= -\mathrm{i}\frac{e}{c}\langle\psi|B\mathcal{V}|\psi\rangle \\ &= -\frac{2\mathrm{i}\hbar^2}{M}\int_0^{2\pi}\mathrm{d}\phi\int_0^\infty \psi^*(r,\phi)\left[\frac{\delta(r)}{r}(\partial_\phi - \mathrm{i}\alpha)\mathrm{e}^{\mathrm{i}\phi}\left(\partial_r + \frac{\mathrm{i}\partial_\phi + \alpha}{r}\right)\right]\psi(r,\phi)r\,\mathrm{d}r\end{aligned} \tag{16}$$

where we have used $\delta^2(\boldsymbol{r}) = \delta(r)/(\pi r)$. With ψ resolved into its angular momentum components,

$$\psi(r,\phi) = \sum_{m=-\infty}^{\infty}\psi_m(r)\mathrm{e}^{\mathrm{i}m\phi} \tag{17}$$

the integrals in (16) are trivially evaluated (note that $\int_0^\infty \delta(r)\,\mathrm{d}r = \frac{1}{2}$), with the result

$$\langle\psi|\mathcal{F}|\psi\rangle = \frac{2\pi\hbar^2}{M}\sum_{m=-\infty}^{\infty}\left[\psi_{m+1}^*(r)(m+1-\alpha)\left(\psi_m'(r) - \frac{(m-\alpha)}{r}\psi_m(r)\right)\right]_{r=0}. \tag{18}$$

Like any vector operator, $\mathcal{F}$ couples only consecutive angular momentum components, m and $m+1$, and, as one would expect, depends only on the behaviour of the wavefunction at the flux line. For well behaved wavefunctions, the leading-order behaviour of $\psi_m(r)$ as $r \to 0$ is given by

$$\psi_m(r) \sim C_m r^{|m-\alpha|}. \tag{19}$$

Sufficient conditions for (19) are discussed in the appendix. Here, we note that (19) ensures that the energy density, $\psi^*(\boldsymbol{r})(H\psi)(\boldsymbol{r})$, is finite at $r = 0$. Substituting (19) into (18), we obtain

$$\langle\psi|\mathcal{F}|\psi\rangle = \frac{2\pi\hbar^2}{M}\sum_{m=-\infty}^{\infty}(m+1-\alpha)(|m-\alpha| - (m-\alpha))C_{m+1}^* C_m r^{|m+1-\alpha|+|m-\alpha|-1}|_{r=0}. \tag{20}$$

The terms in the sum (20) vanish unless $m = a$, where $a = \alpha - \tilde{\alpha}$ denotes the integer part of the flux parameter. Thus, only the 'pinioned' components of the wavefunction, ψ_a and ψ_{a+1}—those whose angular momentum quantum numbers are nearest the flux parameter α—contribute to the force expectation value (18). From (20),

$$\langle\psi|\mathcal{F}|\psi\rangle = \frac{4\pi\hbar^2}{M}\tilde{\alpha}(1-\tilde{\alpha})C_{a+1}^* C_a \tag{21}$$

or, equivalently,

$$\mathcal{F} = \frac{4\pi\hbar^2}{M}\tilde{\alpha}(1-\tilde{\alpha})|\xi_{a+1}\rangle\langle\xi_a| \tag{22}$$

where the state $|\xi_m\rangle$ corresponds to the singular wavefunction

$$\xi_m(r,\phi) = \frac{\delta(r)}{\pi r^{|m-\alpha|+1}}\mathrm{e}^{\mathrm{i}m\phi} \tag{23}$$

so that $\langle\xi_m|\psi\rangle = C_m$. It is readily verified that the force operator (22) transforms according to (5) under the gauge transformation (4). A rigorous derivation of the expectation value (21) for suitably chosen wavefunctions is given in the appendix.

In the preceding derivation, the force operator due to a flux line, like the force for a nonsingular potential, is derived from kinematics, specifically from the commutation relations. The derivation does not require the solution of the Schrödinger equation. Thus, it is straightforward to generalize to the case of more than one flux line (for which solutions of the Schrödinger equation are, in general, not available); the force operator is just a sum of contributions (22) centred around each flux line.

However, to calculate the force on stationary states, or the time dependence of the force on nonstationary states, it is necessary to solve the Schrödinger equation. As is well known, eigenstates of the Aharonov–Bohm Hamiltonian with energy $\hbar^2k^2/2M$ and angular momentum $m\hbar$ are given by

$$\chi_{k,m}(\boldsymbol{r}) = J_{|m-\alpha|}(kr)\mathrm{e}^{\mathrm{i}m\phi} \tag{24}$$

where $J_\nu(z)$ is a Bessel function. From the small-z behaviour, $J_\nu(z) \sim (z/2)^\nu/\Gamma(\nu+1)$, and the reflection formula, $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin\pi\nu$, we obtain from (22) the matrix elements

$$\langle\chi_{p,n}|\mathcal{F}|\chi_{k,m}\rangle = \frac{2\hbar^2}{M}k^{\tilde{\alpha}}p^{1-\tilde{\alpha}}\sin\pi\tilde{\alpha}\,\delta_{m,a}\delta_{n,a+1}. \tag{25}$$

For $k = p$, i.e. for stationary states with the same energy, we obtain the on-shell matrix elements

$$\langle \chi_{k,n}|\mathcal{F}|\chi_{k,m}\rangle = \frac{2\hbar^2}{M} k \sin \pi\tilde{\alpha}\, \delta_{m,a}\delta_{n,a+1} \tag{26}$$

in agreement with results of Nielson and Hedegård (1995) and Shelankov (2000).

A general stationary state $|\Psi\rangle$ is a superposition of eigenstates with k fixed, and may be taken to be of the form

$$|\Psi\rangle = \sum_{m=-\infty}^{\infty} (-\mathrm{i})^{|m-\alpha|} b_m |\chi_{k,m}\rangle. \tag{27}$$

For $b_m = 1$, $\Psi(\boldsymbol{r})$ corresponds to a scattered plane wave incident from the right (Aharonov and Bohm 1959). From (26), we obtain the expectation value

$$\langle \Psi|\mathcal{F}|\Psi\rangle = \frac{2\mathrm{i}\hbar^2}{M} \mathrm{e}^{-\mathrm{i}\pi\tilde{\alpha}} k \sin \pi\tilde{\alpha}\, b^*_{a+1} b_a. \tag{28}$$

Shelankov (1998) has obtained an approximate formula for the transverse component of the force acting on a stationary beam of finite angular width. His analysis is carried out in a singular gauge, in which the vector potential vanishes everywhere except along the y-axis. A stationary beam incident from the right (in fact, Shelankov takes the beam incident from the left, but we revert to the convention of Aharonov and Bohm (1959)) is taken to be of the form $\mathrm{e}^{-\mathrm{i}kx}\psi(x, y)$. Treating ψ_{xx} as small compared to $k\psi_x$ amounts to a paraxial approximation, in which the wave evolves freely in x (with x playing the role of time) for $x \neq 0$, and is scattered by the vector potential at $x = 0$. The change Δp_y in the transverse kinematic momentum, $(\hbar/\mathrm{i}) \int_{-\infty}^{\infty} \psi^*(x, y)\psi_y(x, y)\, \mathrm{d}y$, is then calculated to be

$$\Delta p_y = \hbar \sin 2\pi\alpha \frac{|\psi_{\mathrm{in}}(0)|^2}{\int_{-\infty}^{\infty} |\psi_{\mathrm{in}}(y)|^2\, \mathrm{d}y} \tag{29}$$

where

$$\psi_{\mathrm{in}}(y) = \frac{1}{\sqrt{2\pi k}} \int_{-\infty}^{\infty} a(k_y)\mathrm{e}^{\mathrm{i}k_y y}\, \mathrm{d}k_y \tag{30}$$

is the incident wave at $x = 0^+$, expressed here in terms of its transverse Fourier amplitudes $a(k_y)$. Multiplying Δp_y by the incident flux, which is given paraxially by $(\hbar k/M) \int_{-\infty}^{\infty} |\psi_{\mathrm{in}}(y)|^2\, \mathrm{d}y$, gives *Shelankov's formula* for the transverse force,

$$F_y^{(S)} = \frac{\hbar^2}{M} k \sin 2\pi\tilde{\alpha} |\psi_{\mathrm{in}}(0)|^2. \tag{31}$$

We now show that the y-component of the exact force expectation value, i.e. the imaginary part of (28), coincides with Shelankov's formula (31) in the paraxial regime. (Shelankov (2000) gives the same argument.) As discussed by Berry (1999), the state (27) can alternatively be viewed as a superposition of scattered waves incident from the directions $(\cos\theta, -\sin\theta)$, with amplitudes $A(\theta)$ related to the coefficients b_m according to

$$b_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} A(\theta)\mathrm{e}^{\mathrm{i}(m-\alpha)\theta}\, \mathrm{d}\theta. \tag{32}$$

The paraxial approximation is valid for $A(\theta)$ strongly peaked around $\theta = 0$, with angular width $w \ll 1$. In this case, Berry (1999) has shown that $A(\theta) \sim a(k\theta)$. From (30) and (32), it then follows that $b_m \sim \psi_{\mathrm{in}}((m - \alpha)/k)$ for $|m - \alpha| \ll 1/w$, so that $b^*_{a+1} b_a \sim |\psi_{\mathrm{in}}(0)|^2$.

3. Impulse operator

For nonstationary wavepackets $\psi(\boldsymbol{r})$, whose wavefunctions are not eigenfunctions of the Hamiltonian, the expectation value of the force does not itself have much physical significance. It depends on the behaviour of the wavefunction near the flux line, regardless of where the wavepacket is localized, and can oscillate rapidly as the wavepacket evolves. Of greater physical interest is the impulse imparted to the particle over the course of its evolution, either in the past or future. Let

$$\mathcal{F}(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar}\mathcal{F}\mathrm{e}^{-\mathrm{i}Ht/\hbar} \qquad M\mathcal{V}(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar}M\mathcal{V}\mathrm{e}^{-\mathrm{i}Ht/\hbar} \tag{33}$$

denote the time-evolved force and kinetic momentum operators. Then $M\dot{\mathcal{V}}(t) = \mathcal{F}(t)$. The forward (+) and backward (−) impulse operators are defined by

$$\mathcal{I}_\pm = M\mathcal{V}(\pm\infty) - M\mathcal{V}(0) = \int_0^{\pm\infty} \mathcal{F}(t)\,\mathrm{d}t. \tag{34}$$

Let us compute the kernel of the impulse operator in the position representation, $\mathcal{I}_\pm(\boldsymbol{s},\boldsymbol{r}) = \langle \boldsymbol{s}|\mathcal{I}_\pm|\boldsymbol{r}\rangle$. From the completeness relation,

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty |\chi_{k,m}\rangle\langle\chi_{k,m}|k\,\mathrm{d}k = 1 \tag{35}$$

we obtain

$$\mathcal{I}_\pm(\boldsymbol{s},\boldsymbol{r}) = \frac{1}{4\pi^2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\int_0^{\pm\infty}\mathrm{d}t \times\int_0^\infty\int_0^\infty \langle\boldsymbol{s}|\chi_{p,n}\rangle\langle\chi_{p,n}|\mathcal{F}|\chi_{k,m}\rangle\langle\chi_{k,m}|\boldsymbol{r}\rangle \exp\left(\mathrm{i}\hbar(p^2-k^2)t/2M\right)k\,\mathrm{d}k\,p\,\mathrm{d}p. \tag{36}$$

From the expression (25) for the matrix elements $\langle\chi_{p,n}|\mathcal{F}|\chi_{k,m}\rangle$, the only contribution to the double sum in (36) is from the term $n = a+1$, $m = a$. Substituting the eigenfunctions (24), and letting (s,θ) denote the polar coordinates of $\boldsymbol{s}$, we obtain

$$\mathcal{I}_\pm(\boldsymbol{s},\boldsymbol{r}) = \pm\frac{\hbar^2\sin\pi\tilde{\alpha}}{2\pi^2 M}\exp\left(\mathrm{i}(a+1)\theta - \mathrm{i}a\phi\right)\int_0^\infty\mathrm{d}t \times\int_0^\infty \exp\left(\pm\mathrm{i}\frac{\hbar p^2}{2M}t\right)J_{1-\tilde{\alpha}}(ps)p^{2-\tilde{\alpha}}\,\mathrm{d}p\int_0^\infty\exp\left(\mp\mathrm{i}\frac{\hbar k^2}{2M}t\right)J_{\tilde{\alpha}}(kr)k^{1+\tilde{\alpha}}\,\mathrm{d}k. \tag{37}$$

The k- and p-integrals are of the form (Abramowitz and Stegun 1970)

$$\int_0^\infty \mathrm{e}^{-c^2u^2}J_\nu(bu)u^{\nu+1}\,\mathrm{d}u = \frac{b^\nu}{(2c^2)^{\nu+1}}\mathrm{e}^{-b^2/4c^2}. \tag{38}$$

Substituting this result into (37), we obtain

$$\mathcal{I}_\pm(\boldsymbol{s},\boldsymbol{r}) = \frac{\mathrm{i}M^2}{2\pi^2\hbar}\sin\pi\tilde{\alpha}\,\exp\left(\mathrm{i}(a+1)\theta - \mathrm{i}a\phi \mp \mathrm{i}\pi\tilde{\alpha}\right)s^{1-\tilde{\alpha}}r^{\tilde{\alpha}} \times\int_0^\infty\exp\left(\mp\mathrm{i}\frac{M(r^2-s^2)}{2\hbar t}\right)t^{-3}\,\mathrm{d}t. \tag{39}$$

With the substitution $t = 1/w$, the remaining integral in (39) is of the elementary form

$$\lim_{\epsilon\to 0^+}\int_0^\infty \exp\left(-\frac{\epsilon\pm\mathrm{i}(r^2-s^2)w}{\sigma^2}\right)w\,\mathrm{d}w = \frac{\sigma^4}{(0^+\pm\mathrm{i}(r^2-s^2))^2} \tag{40}$$

with $\sigma^2 = 2\hbar/M$. Substituting (40) into (39), we obtain

$$\mathcal{I}_\pm(\boldsymbol{s}, \boldsymbol{r}) = \frac{2\mathrm{i}\hbar}{\pi^2}\sin\pi\tilde{\alpha}\ \exp\left(\mathrm{i}(a+1)\theta - \mathrm{i}a\phi \mp \mathrm{i}\pi\tilde{\alpha}\right)\frac{r^{\tilde{\alpha}}s^{1-\tilde{\alpha}}}{(0^+ \pm \mathrm{i}(r^2 - s^2))^2}. \quad (41)$$

One can verify that this expression transforms correctly, i.e. according to (5), under the gauge transformation (4).

The denominator in (41) can be alternatively expressed as

$$\frac{1}{(0^+ \pm \mathrm{i}(r^2 - s^2))^2} = P'(1/(r^2 - s^2)) \pm \mathrm{i}\pi\delta'(r^2 - s^2). \quad (42)$$

Here $P'(1/x)$, the derivative of the principal part, acts on functions $f(x)$ according to

$$\int_{-\infty}^{\infty} f(x)P'(1/x)\,\mathrm{d}x = -\int_{-\infty}^{\infty} f'_{\mathrm{odd}}(x)/x\,\mathrm{d}x \quad (43)$$

where $f'_{\mathrm{odd}}(x) = \frac{1}{2}(f'(x) - f'(-x))$ denotes the odd part of $f'(x)$. For subsequent calculations, however, the integral representation (40) will be more convenient.

4. Expectation values of impulse for wavepackets

We parametrize wavepackets by their position $\boldsymbol{R}$, width σ and kinetic momentum $\hbar\boldsymbol{k}$. A convenient form is

$$\psi(\boldsymbol{r}) = \frac{1}{\sigma} f\left(\frac{\boldsymbol{r} - \boldsymbol{R}}{\sigma}\right)\exp\left(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} + \mathrm{i}\alpha\phi\right). \quad (44)$$

Here $f(\boldsymbol{u})$ is a smooth normalized function localized at the origin with unit width and vanishing average (dimensionless) momentum, i.e.

$$\begin{aligned}
&\iint f^* f\,\mathrm{d}^2u = 1 \qquad &&\iint f^* f\boldsymbol{u}\,\mathrm{d}^2u = \boldsymbol{0}\\
&\iint f^* f u^2\,\mathrm{d}^2u = 1 \qquad &&\iint f^*\boldsymbol{\nabla} f\,\mathrm{d}^2u = \boldsymbol{0}.
\end{aligned} \quad (45)$$

We assume that $\sigma \ll R$, so that the wavepacket $\psi(\boldsymbol{r})$ is localized away from the flux line. The phase factor $\exp(\mathrm{i}\alpha\phi)$ in (44) ensures that $\hbar k$ is the kinetic, rather than the canonical, momentum of the wavepacket; its branch is chosen so that the phase factor is continuous over the region where $\psi(\boldsymbol{r})$ is appreciable.

From (41) and (44), the expectation value of the impulse is given by

$$\begin{aligned}
\langle\mathcal{I}_\pm\rangle(\boldsymbol{R}, \boldsymbol{k}, \sigma, \alpha) = \langle\psi|\mathcal{I}_\pm|\psi\rangle = \frac{2\mathrm{i}\hbar}{\pi^2\sigma^2}\mathrm{e}^{\mp\mathrm{i}\pi\tilde{\alpha}}\sin\pi\tilde{\alpha}\\
\times\iiiint f^*\left(\frac{\boldsymbol{s} - \boldsymbol{R}}{\sigma}\right) f\left(\frac{\boldsymbol{r} - \boldsymbol{R}}{\sigma}\right)\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{s})}}{(0^+ \pm \mathrm{i}(r^2 - s^2))^2}\\
\times\mathrm{e}^{\mathrm{i}(1-\tilde{\alpha})\theta + \mathrm{i}\tilde{\alpha}\phi} r^{\tilde{\alpha}}s^{1-\tilde{\alpha}}\,\mathrm{d}^2r\,\mathrm{d}^2s.
\end{aligned} \quad (46)$$

Since f has unit width, the integrand in (46) is appreciable only for $|\boldsymbol{s} - \boldsymbol{R}| \sim \sigma$ and $|\boldsymbol{r} - \boldsymbol{R}| \sim \sigma$. In this region, we can, to leading order in σ/R, replace the phase factors $\exp(\mathrm{i}(1 - \tilde{\alpha})\theta)$ and $\exp(\mathrm{i}\tilde{\alpha}\phi)$ by $\exp(\mathrm{i}(1 - \tilde{\alpha})\Phi)$ and $\exp(\mathrm{i}\tilde{\alpha}\Phi)$, respectively, where Φ is the polar angle of $\boldsymbol{R}$. Likewise, we can replace the factor $r^{\tilde{\alpha}}s^{1-\tilde{\alpha}}$ by R. With the change of variables $\boldsymbol{u} = (\boldsymbol{r} - \boldsymbol{R})/\sigma$ and $\boldsymbol{v} = (\boldsymbol{s} - \boldsymbol{R})/\sigma$ and the integral representation (40), (46) becomes

$$\begin{aligned}
\langle\mathcal{I}_\pm\rangle = \frac{2\mathrm{i}\hbar}{\pi^2}\frac{R}{\sigma^2}\mathrm{e}^{\mathrm{i}\Phi\mp\mathrm{i}\pi\tilde{\alpha}}\sin\pi\tilde{\alpha}\\
\times\int_0^{\infty}\mathrm{d}w\,w\left|\iint f(\boldsymbol{u})\exp\left(\mathrm{i}\sigma\boldsymbol{k}\cdot\boldsymbol{u} \mp 2\mathrm{i}w\frac{\boldsymbol{R}\cdot\boldsymbol{u}}{\sigma} \mp \mathrm{i}wu^2\right)\mathrm{d}^2u\right|^2.
\end{aligned} \quad (47)$$

Thus, the direction of the impulse, $\arg\langle\mathcal{I}_\pm\rangle$, is given by

$$\arg\langle\mathcal{I}_\pm\rangle = \Phi + (\tfrac{1}{2} \mp \tilde{\alpha})\pi. \tag{48}$$

For $\tilde{\alpha} = \frac{1}{2}$, the forward impulse is directed away from the flux line, and the backwards impulse towards the flux line.

There are two parameter regimes where the expression (47) has a simple asymptotic form, namely $k\sigma \ll 1$, which corresponds to slow wavepackets, and $k\sigma \gg R/\sigma$, which corresponds to fast wavepackets. These cases are discussed separately below.

4.1. Slow wavepackets

The condition $k\sigma \ll 1$ implies that the wavepacket spreads (with velocity $\sim \hbar/M\sigma$) more quickly than it moves (with velocity $\hbar k/M$). Since f has unit width, the integrand in (47) is appreciable only for u of order one. In this case, for $k\sigma \ll 1$, the phase factor $\exp(\mathrm{i}\sigma\boldsymbol{k}\cdot\boldsymbol{u})$ is nearly equal to one. On the other hand, the phase factor $\exp(\mp 2\mathrm{i}w\boldsymbol{R}\cdot\boldsymbol{u}/\sigma)$ oscillates rapidly in this region, and hence renders the integral small, unless w is small, of order σ/R. For w of order σ/R, the phase factor $\exp(\mathrm{i}u^2 w)$ is nearly equal to one for u of order one. To leading order in σ/R and $k\sigma$, (47) becomes

$$\begin{aligned}\langle\mathcal{I}_\pm\rangle &= \frac{2\mathrm{i}\hbar}{\pi^2}\frac{R}{\sigma^2}\mathrm{e}^{\mathrm{i}\Phi\mp\mathrm{i}\pi\tilde{\alpha}}\sin\pi\tilde{\alpha}\int_0^\infty \mathrm{d}w\, w\left|\iint f(\boldsymbol{u})\mathrm{e}^{\mp 2\mathrm{i}w\boldsymbol{R}\cdot\boldsymbol{u}/\sigma}\,\mathrm{d}^2u\right| \\ &= 8\mathrm{i}\hbar\frac{R}{\sigma^2}\mathrm{e}^{\mathrm{i}\Phi\mp\mathrm{i}\pi\tilde{\alpha}}\sin\pi\tilde{\alpha}\int_0^\infty\left|\tilde{f}\left(\pm\frac{2\boldsymbol{R}}{\sigma}w\right)\right|^2 w\,\mathrm{d}w\end{aligned} \tag{49}$$

where

$$\tilde{f}(\boldsymbol{\xi}) = \frac{1}{2\pi}\iint f(\boldsymbol{u})\mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{u}}\,\mathrm{d}^2u \tag{50}$$

denotes the normalized Fourier transform of $f(\boldsymbol{u})$. Letting

$$\tilde{\rho}(\hat{e}) = \int_0^\infty |\tilde{f}(\xi\,\hat{e})|^2\xi\,\mathrm{d}\xi \tag{51}$$

denote the probability distribution for the direction, $\hat{e}$, of the dimensionless momentum, $\boldsymbol{\xi} = \xi\,\hat{e}$, we can write

$$\langle\mathcal{I}_\pm\rangle = \frac{2\mathrm{i}\hbar}{R}\mathrm{e}^{\mathrm{i}\Phi\mp\mathrm{i}\pi\tilde{\alpha}}\sin\pi\tilde{\alpha}\,\tilde{\rho}(\pm\hat{\boldsymbol{R}}). \tag{52}$$

Note that if $f(\boldsymbol{u})$ is circularly symmetric, then $\tilde{\rho}(\hat{e})$ is equal to $1/2\pi$.

To leading order in σ/R and $k\sigma$, the impulse (52) is independent of the width and momentum of the wavepacket, and is of order $\hbar/R$ (i.e. inversely proportional to the distance from the flux line). This is much smaller than the dispersion of the momentum, which is of order $\hbar/\sigma$. Therefore, to detect the impulse on slow wavepackets experimentally, one would have to perform a large number of measurements (on the order of $(R/\sigma)^2$) of the asymptotic momentum on an ensemble of identically prepared systems.

By treating the motion of the centre of a slow wavepacket as a classical trajectory, we can derive an expression for the scattering cross-section $\sigma(\theta)$ using the classical formula,

$$\sigma(\theta) = \left|\frac{\mathrm{d}b}{\mathrm{d}\theta}(\theta)\right|. \tag{53}$$

Here b is the impact parameter, and θ is the angular direction of the outgoing trajectory. Consider a slow wavepacket (44) centred on the y-axis at $Y\,\hat{\boldsymbol{y}}$ at $t = 0$ (thus, $R = |Y|$ and $\Phi = \mathrm{sgn}(Y)\pi/2$), moving in the $-\hat{\boldsymbol{x}}$ direction with kinetic momentum $\hbar k$. For simplicity,

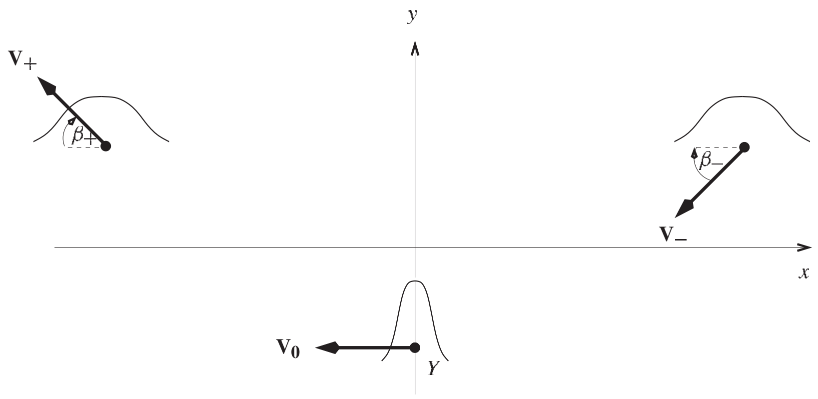

Figure 1. At $t = 0$ the wavepacket is centred at $Y\,\hat{\boldsymbol{y}}$ and moves in the $-\,\hat{\boldsymbol{x}}$ direction. $\beta_\pm$ are the angles of the incoming and outgoing velocities with respect to $-\,\hat{\boldsymbol{x}}$.

we take $f(\boldsymbol{u})$ to be circularly symmetric, so that $\tilde{\rho} = 1/2\pi$. Let β_- denote the angle between the velocities at $t = -\infty$ and 0, and β_+ the angle between the velocities at $t = 0$ and ∞ (see figure 1). From (52), these are given by

$$\cot\beta_\pm = \pm\frac{\mathrm{Re}\,\langle\psi|\mathcal{I}_\pm|\psi\rangle - \hbar k}{\mathrm{Im}\,\langle\psi|\mathcal{I}_\pm|\psi\rangle} = \frac{\sin\pi\tilde{\alpha}\cos\pi\tilde{\alpha} - \pi k Y}{\sin^2\pi\tilde{\alpha}} \tag{54}$$

(thus $\beta_- = \beta_+$). Because the impulse is circularly symmetric, the angles $\beta_\pm$ are unchanged if we rotate the entire system so that the incoming velocity, at $t = -\infty$, is in the $-\,\hat{\boldsymbol{x}}$ direction. In this case, the direction of the outgoing beam is given by

$$\theta = \pi + \beta_+ + \beta_- = \pi + 2\beta_+. \tag{55}$$

To determine the impact parameter b, we appeal to classical angular momentum conservation, $MV_-b = \mathrm{sgn}b\, MV_0R_0$, where V_- is the speed at $t = -\infty$, and R_0 and V_0 are the distance and speed at the point of closest approach to the flux line. For the Aharonov–Bohm Hamiltonian (and, indeed, for any purely magnetic Hamiltonian), the speed $V = \sqrt{\boldsymbol{V}\cdot\boldsymbol{V}}$ is a conserved quantity. Thus $b = \mathrm{sgn}b\, R_0$. We take the point of closest approach to occur at $t = 0$ (when the velocity of the wavepacket is orthogonal to its position), so that $b = Y$. Then, from (53)–(55),

$$\sigma(\theta) = \left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| = \left|\frac{\mathrm{d}\theta}{\mathrm{d}b}\right|^{-1}\left|2\frac{\mathrm{d}\beta_+}{\mathrm{d}Y}\right|^{-1} = \left|2\sin^2\beta_+\frac{\mathrm{d}(\cot\beta_+)}{\mathrm{d}Y}\right|^{-1} = \frac{\sin^2\pi\tilde{\alpha}}{2\pi k\cos^2\theta/2}. \tag{56}$$

Surprisingly, the expression (56) agrees with the exact result found by Aharonov and Bohm (1959). Of course, the preceding should not be regarded as a legitimate derivation of the scattering cross-section. Apart from certain *ad hoc* elements (e.g., circularly symmetric wavepacket and the determination of the impact parameter), a proper derivation of the cross-section from time-dependent solutions of the Schrödinger equation requires wavepackets (unlike the slow ones used here) whose momentum is sharp. Still, it is interesting to ask whether or not this agreement is purely fortuitous.

4.2. Fast wavepackets

A wavepacket initially at a distance R from the flux line with kinetic momentum $\hbar k$ reaches its point of closest approach to the flux line after a time τ of order $R/(\hbar k/M)$. It spreads at a speed W of order $\hbar/M\sigma$. Thus, at closest approach it will have spread a distance of order $W\tau \sim R/k\sigma$. For this to be much less than the width σ, we require $R \ll k\sigma^2$, which is just the condition for fast wavepackets.

Let

$$L_\pm = \left| \iint f(\boldsymbol{u}) \exp\left(\mathrm{i}\sigma\boldsymbol{k}\cdot\boldsymbol{u} \mp 2\mathrm{i}w\boldsymbol{R}\cdot\boldsymbol{u}/\sigma \mp \mathrm{i}wu^2\right) \mathrm{d}^2u \right|^2 \tag{57}$$

denote the $\boldsymbol{u}$-integral which appears in the impulse expectation value (47). Because f is of unit width, the integrand is appreciable only for u of order 1. For $k\sigma \gg R/\sigma$, the phase factor $\exp(\mathrm{i}\sigma\boldsymbol{k}\cdot\boldsymbol{u})$ in (47) is rapidly oscillating, and hence leads to a vanishingly small integral, unless it is balanced by the phase factor $\exp(\mp\mathrm{i}2\mathrm{i}w\boldsymbol{R}\cdot\boldsymbol{u}/\sigma)$. For such a balancing to take place, w must be large, of order $k\sigma^2/R$. Therefore, the quadratic phase factor $\exp(\mp\mathrm{i}wu^2)$ is rapidly oscillating, so (57) can be evaluated using the stationary phase approximation. To leading order in $1/w \sim R/(k\sigma^2)$, we obtain

$$L_\pm = \frac{\pi^2}{w^2}\left| f\left(\frac{\mp\sigma^2\boldsymbol{k}/2w - \boldsymbol{R}}{\sigma}\right)\right|^2. \tag{58}$$

Substituting into (47), we obtain

$$\begin{aligned}\langle \mathcal{I}_\pm \rangle &= 2\mathrm{i}\hbar\frac{R}{\sigma^2}\mathrm{e}^{\mathrm{i}\Phi\mp\mathrm{i}\pi\tilde{\alpha}}\sin\pi\tilde{\alpha}\int_0^\infty \left| f\left(\frac{\mp\sigma^2\boldsymbol{k}/2w - \boldsymbol{R}}{\sigma}\right)\right|^2 \frac{\mathrm{d}w}{w} \\ &= 2\mathrm{i}\hbar R\mathrm{e}^{\mathrm{i}\Phi\mp\mathrm{i}\pi\tilde{\alpha}}\sin\pi\tilde{\alpha}\int_0^\infty |\psi(\mp r\,\hat{\boldsymbol{k}})|^2\frac{\mathrm{d}r}{r}\end{aligned} \tag{59}$$

where we have used (44) to express the integral in terms of the wavefunction $\psi(\boldsymbol{r})$. Note that $\psi(\boldsymbol{r})$ behaves for small r as $r^{\tilde{\alpha}}$ or $r^{1-\tilde{\alpha}}$ (cf (19)), so that the integral in (59) is convergent.

In what follows, let us assume for concreteness that $\boldsymbol{k}$ is directed along $-\hat{\boldsymbol{x}}$, so that the wavepacket is moving to the left. We write $\boldsymbol{R} = X\,\hat{\boldsymbol{x}} + Y\,\hat{\boldsymbol{y}}$. Unless the wavepacket is centred near the x-axis (specifically, unless $|Y| \sim \sigma$), $\psi(\mp r\,\hat{\boldsymbol{k}})$ will be negligible over the range of integration in (59). Thus, to leading order in σ/R, we may take $R\mathrm{e}^{\mathrm{i}\Phi} = X$. Substituting this result into (59), we obtain the expression

$$\langle \mathcal{I}_\pm \rangle = 2\mathrm{i}\hbar\mathrm{e}^{\mp\mathrm{i}\pi\tilde{\alpha}}X\sin\pi\tilde{\alpha}\int_0^{\pm\infty}|\psi(x,0)|^2\frac{\mathrm{d}x}{x} \tag{60}$$

for the impulse.

Since the wavepacket is centred near $X\,\hat{\boldsymbol{x}}$, the x-integral in (60) is negligible unless $X > 0$ in the forward (+) case (so that the wavepacket is moving towards the flux line), or unless $X < 0$ in the backward (−) case (so that the wavepacket is moving away from the flux line). Assuming that $\pm X > 0$, the main contribution to the integral comes from $|x - X| \sim \sigma$, so that, to leading order in σ/R, we can replace the factor $1/x$ by $1/X$ in (60), and extend the lower limit of the x integral to $\mp\infty$. Letting

$$P_{\mathrm{trans}}(y) = \int_{-\infty}^{\infty}|\psi(x,y)|^2\,\mathrm{d}x \tag{61}$$

denote the wavepacket's probability density in y (the direction transverse to the incident velocity), we obtain, to leading order in σ/R and $R/(k\sigma^2)$, the expression

$$\langle \mathcal{I}_\pm \rangle = \pm 2\mathrm{i}\hbar\mathrm{e}^{\mp\mathrm{i}\pi\tilde{\alpha}}\sin\pi\tilde{\alpha}\,\Theta(\pm X)\,P_{\mathrm{trans}}(0) \tag{62}$$

for the impulse on fast wavepackets. Here $\Theta(x)$ is the unit step function.

The impulse (62) is independent of the wavenumber k. To leading order, it vanishes for wavepackets which miss the flux line (e.g., $|Y| \gg \sigma$, or $\pm X > 0$), as shown previously by

Olariu and Popescu (1983, 1985). For fast wavepackets which hit the flux line, taking $P_{\mathrm{trans}}(Y)$ to be of order $1/\sigma$ for $|Y| \sim \sigma$, we obtain that the impulse is of order $\hbar/\sigma$. Therefore, it is of the same order as the dispersion in momentum, in contrast with the case of slow wavepackets, for which the impulse is much smaller (by a factor of σ/R) than the dispersion.

The y-component of the forward impulse, i.e. the imaginary part of (62), is given in the forward case by

$$\langle\psi|I_{+y}|\psi\rangle = \pm\hbar \sin 2\pi\tilde{\alpha}\, \Theta(\pm X) P_{\mathrm{trans}}(y). \tag{63}$$

This can be regarded as an analogue in the time domain of Shelankov's formula (29) for the transverse momentum imparted to a stationary paraxial beam.

5. Enclosed and distributed fluxes

Two well known regularizations of the Aharonov–Bohm flux line are to enclose the flux in an impenetrable cylindrical barrier, or to distribute the flux uniformly in a cylindrical tube. Here we show that the force and impulse operators in both cases approach the Aharonov–Bohm limit, in a sense to be explained, as the radius ϵ of the cylinder approaches zero.

In a circularly symmetric gauge, the vector potential for both models is of the form $\boldsymbol{A}^\epsilon(\boldsymbol{r}) = A^\epsilon(r)\,\hat{\phi}$. The kinetic momentum is given by

$$M\mathcal{V}^\epsilon = M(V_x^\epsilon + \mathrm{i}V_y^\epsilon) = \frac{\hbar}{\mathrm{i}}\mathrm{e}^{\mathrm{i}\phi}\left(\partial_r + \frac{\mathrm{i}\partial_\phi}{r} + \frac{2\pi}{\Phi_0}A^\epsilon(r)\right) \tag{64}$$

and the regularized Hamiltonian by

$$H^\epsilon = \frac{1}{2}M((V_x^\epsilon)^2 + (V_y^\epsilon)^2) = -\frac{\hbar^2}{2M}\left(\partial_r^2 + \frac{\partial_r}{r} + \left(\frac{\mathrm{i}\partial_\phi}{r} + \frac{2\pi}{\Phi_0}A^\epsilon(r)\right)^2\right). \tag{65}$$

The eigenfunctions of the Hamiltonian and kinetic angular momentum, with energy $E = \hbar^2k^2/2M$ and kinetic angular momentum $m\hbar$, are of the form

$$\chi_{k,m}^\epsilon(\boldsymbol{r}) = R_{k,m}^\epsilon(r)\mathrm{e}^{\mathrm{i}m\phi}. \tag{66}$$

The radial eigenfunctions $R_{k,m}^\epsilon(r)$ are taken to be real and normalized, like the Aharonov–Bohm radial eigenfunctions $J_{|m-\alpha|}(kr)$, according to

$$\int_0^\infty R_{p,m}^\epsilon(r)R_{k,m}^\epsilon(r)r\,\mathrm{d}r = \frac{\delta(k-p)}{k}. \tag{67}$$

These conditions determine the radial eigenfunctions up to an overall sign, which is fixed by requiring that $R_{k,m}^\epsilon(r)$ approach $J_{|m-\alpha|}(kr)$ as ϵ approaches zero.

Let $c_m(k)$ denote a smooth, normalized probability amplitude for energy and angular momentum. Let $\psi(\boldsymbol{r})$ and $\psi^\epsilon(\boldsymbol{r})$ denote the corresponding coordinate wavefunctions for the Aharonov–Bohm and regularized Hamiltonians, respectively. That is,

$$\psi(\boldsymbol{r}) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty c_m(k)J_{|m-\alpha|}(kr)\mathrm{e}^{\mathrm{i}m\phi}k\,\mathrm{d}k \tag{68}$$

$$\psi^\epsilon(\boldsymbol{r}) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty c_m(k)R_{k,m}^\epsilon(r)\mathrm{e}^{\mathrm{i}m\phi}k\,\mathrm{d}k. \tag{69}$$

From the preceding discussion, it is clear that $\psi^\epsilon(\boldsymbol{r})$ approaches $\psi(\boldsymbol{r})$ pointwise as ϵ approaches zero. It turns out that their force and impulse expectation values also coincide as $\epsilon \to 0$, i.e.

$$\lim_{\epsilon\to 0}\langle\psi^\epsilon|\mathcal{F}^\epsilon|\psi^\epsilon\rangle = \langle\psi|\mathcal{F}|\psi\rangle \tag{70}$$

$$\lim_{\epsilon\to 0}\langle\psi^\epsilon|\mathcal{I}_\pm^\epsilon|\psi^\epsilon\rangle = \langle\psi|\mathcal{I}_\pm|\psi\rangle. \tag{71}$$

Note that (70) and (71) do not imply, nor is it the case, that the operators $\mathcal{F}^\epsilon$ and $\mathcal{I}_\pm^\epsilon$ approach their Aharonov–Bohm counterparts, $\mathcal{F}$ and $\mathcal{I}_\pm$, as ϵ approaches 0. Indeed, neither does the regularized Hamiltonian H^ϵ approach the Aharonov–Bohm Hamiltonian H; given $\epsilon > 0$, one can construct wavefunctions whose energy expectation values with respect to H^ϵ and H differ by arbitrarily large amounts.

Instead of (70) and (71), we show below, for the enclosed and distributed fluxes separately, that the eigenstate matrix elements of the regularized force operator approach the Aharonov–Bohm limit as $\epsilon \to 0$, i.e.

$$\lim_{\epsilon\to 0}\langle \chi^\epsilon_{p,n}|\mathcal{F}^\epsilon|\chi^\epsilon_{k,m}\rangle = \langle \chi_{p,n}|\mathcal{F}|\chi_{k,m}\rangle = \frac{2\hbar^2}{M}\sin\pi\tilde{\alpha}k^{\tilde{\alpha}}p^{1-\tilde{\alpha}}\delta_{m,a}\delta_{n,a+1}. \tag{72}$$

Formally, of course, (70) and (72) are equivalent. However, for the sake of brevity we shall omit the details required for a rigorous demonstration. These details are straightforward to supply, and are similar to those given in the main part of the appendix.

The result (71) for the impulse follows from the corresponding result (70) for the force, once it has been established that the force expectation values $\langle\psi^\epsilon|\mathcal{F}^\epsilon(t)|\psi^\epsilon\rangle$ and $\langle\psi|\mathcal{F}(t)|\psi\rangle$ are integrable in time. In the appendix it is shown that, in the Aharonov–Bohm case, the force expectation value decays as $1/t^2$; a similar argument may be given for the regularized force.

5.1. *Enclosed flux*

The kinetic momentum $M\mathcal{V}^\epsilon$ and Hamiltonian H^ϵ have the same operational form as in the Aharonov–Bohm case, but act on wavefunctions defined on $r \geqslant \epsilon$ which vanish at $r = \epsilon$. For two such wavefunctions, $\psi^\epsilon(\boldsymbol{r})$ and $\eta^\epsilon(\boldsymbol{r})$, assumed to be smooth and normalized, we have

$$\begin{aligned}\langle\psi^\epsilon|\mathcal{F}^\epsilon|\eta^\epsilon\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\langle\psi^\epsilon|M\mathcal{V}^\epsilon\eta^\epsilon\rangle = \frac{\mathrm{i}}{\hbar}[\langle H^\epsilon\psi^\epsilon|M\mathcal{V}^\epsilon\eta^\epsilon\rangle - \langle\psi^\epsilon|M\mathcal{V}^\epsilon(H^\epsilon\eta^\epsilon)\rangle]\\ &= \frac{\mathrm{i}}{\hbar}(\langle H^\epsilon\psi^\epsilon|M\mathcal{V}^\epsilon\eta^\epsilon\rangle - \langle\psi^\epsilon|H^\epsilon(M\mathcal{V}^\epsilon\eta^\epsilon)\rangle).\end{aligned} \tag{73}$$

The last equality follows from the fact that the commutator $[H^\epsilon, M\mathcal{V}^\epsilon]$ is proportional to the Lorentz force operator (1), which vanishes for the enclosed flux. However, the final expression in (73) does not vanish; the relation $\langle H^\epsilon\psi^\epsilon|\xi^\epsilon\rangle = \langle\psi^\epsilon|H^\epsilon\xi^\epsilon\rangle$, where

$$\xi^\epsilon(\boldsymbol{r}) = (M\mathcal{V}^\epsilon\eta^\epsilon)(\boldsymbol{r}) \tag{74}$$

need not hold, because $\xi^\epsilon(\boldsymbol{r})$ need not vanish at $r = \epsilon$ (alternatively, $|\xi^\epsilon\rangle$ is not in the domain of H^ϵ). Indeed, integration by parts in (65) gives

$$\begin{aligned}\langle H^\epsilon\psi^\epsilon|\xi^\epsilon\rangle - \langle\psi^\epsilon|H^\epsilon\xi^\epsilon\rangle &= -\frac{\hbar^2}{2M}\int_0^{2\pi}\int_\epsilon^\infty\Bigg[\left(\psi^{\epsilon *}_{rr} + \frac{\psi^{\epsilon *}_r}{r} + \frac{(\mathrm{i}\psi^{\epsilon *}_\phi + \alpha\psi^{\epsilon *})}{r^2}\right)\xi^\epsilon\\ &\quad -\psi^{\epsilon *}\left(\xi^\epsilon_{rr} + \frac{\xi^\epsilon_r}{r} + \frac{(\mathrm{i}\xi^\epsilon_\phi + \alpha\xi^\epsilon)^2}{r^2}\right)\Bigg]r\,\mathrm{d}r\,\mathrm{d}\phi\\ &= -\frac{\hbar^2}{2M}\int_0^{2\pi}\psi^{\epsilon *}_r(\epsilon,\phi)\xi^\epsilon(\epsilon,\phi)\,\mathrm{d}\phi.\end{aligned} \tag{75}$$

From (64) and (74),

$$\xi^\epsilon(\epsilon,\phi) = \frac{\hbar}{\mathrm{i}}\mathrm{e}^{\mathrm{i}\phi}\eta_r(\epsilon,\phi). \tag{76}$$

Substituting this result into (75), we obtain

$$\langle\psi^\epsilon|\mathcal{F}^\epsilon|\eta^\epsilon\rangle = \frac{\hbar^2}{2M}\epsilon\int_0^{2\pi}\psi^{\epsilon *}_r(\epsilon,\phi)\eta^{\epsilon *}_r(\epsilon,\phi)\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi \tag{77}$$

a result obtained previously by Peshkin (1981, 1989). Note that if we were to substitute, for $\psi^\epsilon(r)$ and $\eta^\epsilon(r)$, the leading-order behaviour (19) of Aharonov–Bohm wavefunctions, we would recover, formally, the Aharonov–Bohm result (21) for the force expectation value.

Instead, we take $\psi^\epsilon(\boldsymbol{r})$ and $\eta^\epsilon(\boldsymbol{r})$ in (77) to be eigenfunctions of the regularized Hamiltonian. Then

$$\langle \chi^\epsilon_{p,n}|\mathcal{F}^\epsilon|\chi^\epsilon_{k,m}\rangle = \frac{\pi\hbar^2}{M}\epsilon R^\epsilon_{p,m+1}{}'(\epsilon)R^\epsilon_{k,m}{}'(\epsilon)\delta_{n,m+1}. \tag{78}$$

(Strictly speaking, this is not legitimate, as there would appear boundary terms at $r = \infty$ in (75). However, these would vanish when we consider expectation values, as in (70).)

To evaluate (78) we need the derivatives of the radial eigenfunctions at $r = \epsilon$. The radial wavefunctions themselves are given by

$$R^\epsilon_{k,m}(r) = C^\epsilon_{k,m}(N_{|m-\alpha|}(k\epsilon)J_{|m-\alpha|}(kr) - J_{|m-\alpha|}(k\epsilon)N_{|m-\alpha|}(kr)) \tag{79}$$

where $N_\nu(z)$ is the Neumann function. The constant $C^\epsilon_{k,m}$ is determined by the normalization condition (67), and is given by

$$C^\epsilon_{k,m} = (J^2_{|m-\alpha|}(k\epsilon) + N^2_{|m-\alpha|}(k\epsilon))^{-\frac{1}{2}} \tag{80}$$

and, to leading order in ϵ, by

$$C^\epsilon_{k,m} = |N_{m-\alpha}(k\epsilon)|^{-1} = \frac{\pi}{\Gamma(|m-\alpha|)}\left(\frac{k\epsilon}{2}\right)^{|m-\alpha|}. \tag{81}$$

The Wronskian relation, $J_\nu(z)N'_\nu(z) - J'_\nu(z)N_\nu(z) = 2/(\pi z)$, implies that

$$R^\epsilon_{k,m}{}'(\epsilon) = -\frac{1}{\pi(\epsilon/2)}C^\epsilon_{k,m} \qquad R^\epsilon_{p,m+1}{}'(\epsilon) = -\frac{1}{\pi(\epsilon/2)}C^\epsilon_{p,m+1}. \tag{82}$$

Substituting (81) and (82) into (78), we obtain, to leading order in ϵ,

$$\langle \chi^\epsilon_{p,n}|\mathcal{F}^\epsilon|\chi^\epsilon_{k,m}\rangle = \frac{2\pi\hbar^2}{M}\frac{k^{|m-\alpha|}p^{|m+1-\alpha|}}{\Gamma(|m+1-\alpha|)\Gamma(|m-\alpha|)}\left(\frac{\epsilon}{2}\right)^{|m+1-\alpha|+|m-\alpha|-1}\delta_{m,n+1}. \tag{83}$$

In the limit $\epsilon \to 0$, only the $m = a$ term survives, and the reflection formula for the Γ-function gives

$$\begin{aligned}\lim_{\epsilon\to 0}\langle \chi^\epsilon_{p,n}|\mathcal{F}^\epsilon|\chi^\epsilon_{k,m}\rangle &= \frac{2\pi\hbar^2}{M}\frac{k^{\tilde\alpha}p^{1-\tilde\alpha}}{\Gamma(1-\tilde\alpha)\Gamma(\tilde\alpha)}\delta_{m,a}\delta_{n,a+1}\\ &= \frac{2\hbar^2}{M}\sin\pi\tilde\alpha\, k^{\tilde\alpha}p^{1-\tilde\alpha}\delta_{m,a}\delta_{n,a+1}\end{aligned} \tag{84}$$

in accord with (72).

5.2. Distributed flux

The distributed flux model was used by Nielsen and Hedegård (1995) to obtain, from the force balance equations, the on-shell matrix elements of the force in the limit $\epsilon \to 0$. Here we carry out a different calculation to obtain the general matrix elements of the force.

It suffices to consider the case $\alpha > 0$ (the case of negative flux is obtained from time-reversal). The vector potential is given by

$$A^\epsilon(r) = \alpha\Phi_0 r/(2\pi\epsilon^2) \qquad r < \epsilon \tag{85}$$

$$= \alpha\Phi_0/2\pi r \qquad r \geqslant \epsilon \tag{86}$$

corresponding to the magnetic field $B^\epsilon(r) = (\alpha\Phi_0/\pi\epsilon^2)\Theta(\epsilon - r)$, where $\Theta(x)$ is the unit step function. In this case, the force operator is just the Lorentz force (1). It is convenient to introduce the dimensionless radial coordinate $u = r^2/\epsilon^2$, so that the interior of the flux tube is given by $0 \leqslant u \leqslant 1$. The kinetic momentum is given by

$$M\mathcal{V}^\epsilon = \frac{\hbar}{\mathrm{i}} \mathrm{e}^{\mathrm{i}\phi} \frac{u^{1/2}}{\epsilon} \left(2\partial_u + \frac{\mathrm{i}\partial_\phi}{u} + \alpha \right). \tag{87}$$

Then

$$\begin{aligned} \mathcal{F} &= -\mathrm{i} \frac{e}{2Mc} (M\mathcal{V}^\epsilon B^\epsilon + B^\epsilon M\mathcal{V}^\epsilon) \\ &= -\frac{2\hbar^2}{M\epsilon^3} \alpha \mathrm{e}^{\mathrm{i}\phi} u^{\frac{1}{2}} \left[\Theta(1-u) \left(2\partial_u + \mathrm{i}\frac{\partial_\phi}{u} + \alpha \right) - \delta(u-1) \right]. \end{aligned} \tag{88}$$

The matrix elements of the force are given by

$$\begin{aligned} \langle \chi^\epsilon_{p,n} | \mathcal{F}^\epsilon | \chi^\epsilon_{k,m} \rangle &= -2\pi \frac{\hbar^2}{M\epsilon} \alpha \delta_{n,m+1} \\ &\quad \times \left[\int_0^1 T^\epsilon_{p,m+1} \left(2{T^\epsilon_{k,m}}' + \left(\alpha - \frac{m}{u} \right) T^\epsilon_{k,m} \right) u^{\frac{1}{2}} \,\mathrm{d}u - t^\epsilon_{p,m+1} t^\epsilon_{k,m} \right] \end{aligned} \tag{89}$$

where $T^\epsilon_{k,m}(u)$ denotes the radial eigenfunction expressed in terms of the scaled variable u, and $t^\epsilon_{k,m} = T^\epsilon_{k,m}(1)$.

Inside the flux tube, the radial eigenfunctions are given by Landau and Lifshitz (1965)

$$T^\epsilon_{k,m}(u) = C^\epsilon_{k,m} \mathrm{e}^{-\alpha u/2} u^{|m|/2} M \left(-\frac{(k\epsilon)^2}{4\alpha} + \frac{|m| - m + 1}{2}, |m| + 1, \alpha u \right) \qquad 0 \leqslant u \leqslant 1 \tag{90}$$

where $M(a, b, z)$ is the confluent hypergeometric function (Abramowitz and Stegun 1970). Outside the flux tube,

$$R^\epsilon_{k,m}(r) = D^\epsilon_{k,m} J_{|m-\alpha|}(kr) + E^\epsilon_{k,m} N_{|m-\alpha|}(kr) \qquad r \geqslant \epsilon. \tag{91}$$

The coefficients $C^\epsilon_{k,m}$, $D^\epsilon_{k,m}$ and $E^\epsilon_{k,m}$ are determined by requiring the radial eigenfunction and its first derivative to be continuous at $r = \epsilon$ (the second derivative is then continuous there as well, as it turns out), and by the normalization condition

$$(D^\epsilon_{k,m})^2 + (E^\epsilon_{k,m})^2 = 1 \tag{92}$$

which follows from (67).

To evaluate the force matrix element (89), we only require the function inside the flux cylinder. Straightforward algebra gives the coefficient $C^\epsilon_{k,m}$, to leading order in ϵ, as

$$C^\epsilon_{k,m} = \frac{2\mathrm{e}^{\alpha/2} (\frac{1}{2}k\epsilon)^{|m-\alpha|}}{\Gamma(|m-\alpha|)[(|m-\alpha| + |m| - \alpha) f_m + 2f'_m]} \tag{93}$$

where

$$F_m(u) = M(\tfrac{1}{2}(|m| - m + 1), |m| + 1, \alpha u) \tag{94}$$

and f_m and f'_m denote the values of F_m and F'_m at $u = 1$.

Substituting (90) and (93) into (89), we find that $\langle \chi^\epsilon_{p,n} | \mathcal{F}^\epsilon | \chi^\epsilon_{k,m} \rangle$ is of order $\epsilon^{|m-\alpha|+|m+1-\alpha|-1}$, and therefore vanishes in the limit $\epsilon \to 0$ unless $m = a$. We obtain

$$\lim_{\epsilon \to 0} \langle \chi^\epsilon_{p,n} | \mathcal{F}^\epsilon | \chi^\epsilon_{k,m} \rangle = \frac{2\hbar^2}{M} \sin \pi\tilde{\alpha}\, k^{\tilde{\alpha}} p^{1-\tilde{\alpha}} \frac{L(\alpha)}{R(\alpha)} \delta_{m,a} \delta_{n,a+1} \tag{95}$$

where

$$\begin{aligned} L(\alpha) &= -2\alpha \int_0^1 F'_a F_{a+1} \mathrm{e}^{\alpha(1-u)} u^{a+1} \,\mathrm{d}u + \alpha f_a f_{a+1} \\ R(\alpha) &= 2(p(1) f_{a+1} + f'_{a+1} f'_a) \end{aligned} \tag{96}$$

and

$$p(u) = a + 1 - \alpha u. \tag{97}$$

As we show below, $L(\alpha) = R(\alpha)$, or, equivalently,

$$\int_0^1 2\alpha F_a' F_{a+1} \mathrm{e}^{\alpha(1-u)} u^{a+1}\, \mathrm{d}u = \alpha f_a f_{a+1} - 2(p(1) f_{a+1} + f_{a+1}') f_a'. \tag{98}$$

With this identity, (95) gives the required result (72).

To establish the identity (98), it is convenient to express F_{a+1} in terms of F_a by means of the recurrence relation (Abramowitz and Stegun 1970)

$$(a + \tfrac{1}{2}) M(\tfrac{1}{2}, a+2, u) = (a+1)(M(\tfrac{1}{2}, a+1, u) - M'(\tfrac{1}{2}, a+1, u)) \tag{99}$$

which implies that

$$(a + \tfrac{1}{2}) F_{a+1} = (a+1)(F_a - F_a'/\alpha). \tag{100}$$

With the differential equation

$$u F_a'' = -p(u) F_a' + \frac{\alpha}{2} F_a \tag{101}$$

it is straightforward to show that the integrand on the left-hand side of (98) is given by $W'(u)$, where

$$W(u) = 2\frac{a+1}{a+\frac{1}{2}} u^{a+1} \mathrm{e}^{\alpha(1-u)} \left(\frac{\alpha}{2} F_a^2 - p(u) F_a F_a' - u F_a'^2 \right). \tag{102}$$

$W(0)$ vanishes, whereas $W(1)$, with the aid of (100) and (101), is seen to be equal to the right-hand side of (98).

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Appendix. Wavepacket expectation values

The force and impulse due to an Aharonov–Bohm flux line can be calculated rigorously for suitably well behaved wavefunctions $\psi(\boldsymbol{r})$. We will take these to be such that

$c_m(k) = \langle \chi_{k,m} | \psi \rangle$ is smooth in k and falls off, along with its derivatives, faster than any power of k and m. (A.1)

Using standard arguments, one can show that (A.1) implies the following properties of $\psi(\boldsymbol{r})$ and $(H\psi)(\boldsymbol{r})$, where H is the Aharonov–Bohm Hamiltonian:

$\psi(\boldsymbol{r})$ and $(H\psi)(\boldsymbol{r})$ are smooth for $r > 0$ and fall off, along with their derivatives, faster than any power of r (A.2)

and

$$\begin{aligned} \psi_m(r) &= C_m r^{|m-\alpha|} + \mathrm{O}(r^{|m-\alpha|+1}) \\ (H\psi)_m(r) &= D_m r^{|m-\alpha|} + \mathrm{O}(r^{|m-\alpha|+1}) \end{aligned} \tag{A.3}$$

where, in general,

$$\eta_m(r) = \frac{1}{2\pi} \int_0^{2\pi} \eta(r, \phi) \mathrm{e}^{-\mathrm{i}m\phi}\, \mathrm{d}\phi. \tag{A.4}$$

(In fact, properties (A.2) and (A.3) are also shared by $(H^j\psi)(\boldsymbol{r})$, for $j > 1$. The argument to follow would hold under weaker conditions, but we assume (A.1) for simplicity.)

The expectation value of force is given by

$$\begin{aligned}\langle\psi|\mathcal{F}|\psi\rangle &= \iint (\dot{\psi}^*(M\mathcal{V}\psi) + \psi^*(M\mathcal{V}\dot{\psi}))\,\mathrm{d}^2r \\ &= \frac{1}{\mathrm{i}\hbar}\iint (-(H\psi^*)(M\mathcal{V}\psi) + \psi^*(M\mathcal{V}H\psi))\,\mathrm{d}^2r \end{aligned} \tag{A.5}$$

where $M\mathcal{V}$ is given by (15). From (A.2) and (A.3), it is evident that the $\boldsymbol{r}$-integral in (A.5) converges absolutely. This allows us to introduce a factor $\exp(-\epsilon^2r^2)$ in the integrand, and then take the limit of the integral as $\epsilon \to 0$. This Gaussian factor will justify subsequent reorderings of operations. Note that the integral cannot be expressed in terms of the expectation value of the commutator $[H, M\mathcal{V}]$, because of the singularity in the radial derivative of $\psi(\boldsymbol{r})$ at the origin (specifically, $(M\mathcal{V}\psi)(\boldsymbol{r})$ is not in the domain of H).

We introduce the eigenfunction expansion

$$\psi(\boldsymbol{r}) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty c_m(k)J_{|m-\alpha|}(kr)\mathrm{e}^{\mathrm{i}m\phi}k\,\mathrm{d}k \tag{A.6}$$

and a similar expansion for $(H\psi)(\boldsymbol{r})$, with $c_m(k)$ replaced by $-(\hbar^2k^2/2M)c_m(k)$. Using standard arguments, one can show that (A.1) implies that the differential operator $M\mathcal{V}$, when applied to ψ and $H\psi$, can be taken inside the m-sum and k-integral. The recurrence relation,

$$J_{\nu\pm1}(z) = \mp\left(J'_\nu(z) \mp \frac{\nu}{z}J_\nu(z)\right) \tag{A.7}$$

implies that

$$M\mathcal{V}(J_{|m-\alpha|}(kr)\mathrm{e}^{\mathrm{i}m\phi}) = \begin{cases}\mathrm{sgn}(m-a)\mathrm{i}\hbar kJ_{|m+1-\alpha|}(kr)\mathrm{e}^{\mathrm{i}(m+1)\phi} & m\neq a\\ -\mathrm{i}\hbar kJ_{\tilde{\alpha}-1}(kr)\mathrm{e}^{\mathrm{i}(a+1)\phi} & m=a.\end{cases} \tag{A.8}$$

Substituting (A.6) and (A.8) into (A.5), along with the eigenfunction expansion of $\psi^*(\boldsymbol{r})$ with coefficients $c_n^*(p)$, we obtain

$$\begin{aligned}\langle\psi|\mathcal{F}|\psi\rangle = \lim_{\epsilon\to0}\frac{\hbar^2}{8\pi^2M}&\int_0^\infty \mathrm{e}^{-\epsilon^2r^2}r\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\phi\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(m+1-n)\phi}\int_0^\infty\mathrm{d}p\int_0^\infty\mathrm{d}k\\ &\times c_n^*(p)c_m(k)k^2p(k^2-p^2)J_{|n-\alpha|}(pr)\\ &\times\begin{cases}\mathrm{sgn}(m-a)J_{|m+1-\alpha|}(kr) & m\neq a\\ -J_{\tilde{\alpha}-1}(kr) & m=a.\end{cases}\end{aligned} \tag{A.9}$$

The sums and integrals in (A.9) are uniformly and absolutely convergent, and can be interchanged. On performing the ϕ-integral, the sum on n collapses to the single term $n = m+1$. We obtain

$$\langle\psi|\mathcal{F}|\psi\rangle = \frac{\hbar^2}{4\pi M}\lim_{\epsilon\to0}\sum_{m=-\infty}^{\infty}\int_0^\infty\int_0^\infty K_m^\epsilon(k,p)c_{m+1}^*(p)c_m(k)\,\mathrm{d}k\,\mathrm{d}p \tag{A.10}$$

where, for $m \neq a$,

$$K_m^\epsilon(k,p) = \mathrm{sgn}(m-a)k^2p(k^2-p^2)\int_0^\infty \mathrm{e}^{-\epsilon^2r^2}J_\nu(pr)J_\nu(kr)r\,\mathrm{d}r \qquad \nu = |m-\alpha+1| \tag{A.11}$$

and, for $m = a$,

$$K_a^\epsilon(k,p) = k^2p(p^2-k^2)\int_0^\infty \mathrm{e}^{-\epsilon^2r^2}J_\nu(pr)J_{-\nu}(kr)r\,\mathrm{d}r \qquad \nu = 1-\tilde{\alpha}. \tag{A.12}$$

Below, in appendix A.1, we show that the contributions from the $m \neq a$ terms vanish in the limit, i.e.

$$\lim_{\epsilon\to 0}\sum_{m\neq a}\int_0^\infty\int_0^\infty K_m^\epsilon(k,p)c_{a+1}^*(p)c_a(k)\,\mathrm{d}k\,\mathrm{d}p = 0 \tag{A.13}$$

while, in appendix A.2, we show for the $m = a$ term that

$$\lim_{\epsilon\to 0}\int_0^\infty\int_0^\infty K_a^\epsilon(k,p)c_{a+1}^*(p)c_a(k)\,\mathrm{d}k\,\mathrm{d}p = \frac{2}{\pi}\sin\pi\tilde{\alpha}\int_0^\infty\int_0^\infty k^{1+\tilde{\alpha}}p^{2-\tilde{\alpha}}c_{a+1}^*(p)c_a(k)\,\mathrm{d}k\,\mathrm{d}p. \tag{A.14}$$

Substitution of (A.13) and (A.14) into (A.10) gives

$$\langle\psi|\mathcal{F}|\psi\rangle = \frac{\hbar^2}{2\pi^2 M}\sin\pi\tilde{\alpha}\int_0^\infty\int_0^\infty k^{1+\tilde{\alpha}}p^{2-\tilde{\alpha}}c_{a+1}^*(p)c_a(k)\,\mathrm{d}k\,\mathrm{d}p. \tag{A.15}$$

This is equivalent to the matrix element (25), obtained formally in section 2.

We note that, with $\epsilon = 0$, the integrals in (A.11) and (A.12) correspond to singular (i.e. not absolutely convergent) cases of the discontinuous Weber–Schafheitlin integral (Abramowitz and Stegun 1970). Formal evaluation of these integrals would give (A.13) and (A.14) immediately. The arguments in appendix A.1 and A.2 serve to justify these formal results.

To obtain the force expectation value (21), we express $c_a(k)$ and $c_{a+1}^*(p)$ in (A.15) in terms of $\psi(\boldsymbol{r})$ to obtain

$$\langle\psi|\mathcal{F}|\psi\rangle = \frac{2\hbar^2}{M}\sin\pi\tilde{\alpha}\lim_{\epsilon\to 0}\int_0^\infty\int_0^\infty k^{1+\tilde{\alpha}}p^{2-\tilde{\alpha}}\mathrm{e}^{-\epsilon^2(k^2+p^2)} \times\left(\int_0^\infty\psi_{a+1}^*(s)J_{1-\tilde{\alpha}}(ps)s\,\mathrm{d}s\right)\left(\int_0^\infty\psi_a(r)J_{\tilde{\alpha}}(kr)r\,\mathrm{d}r\right)\mathrm{d}k\,\mathrm{d}p. \tag{A.16}$$

Note that the convergence factor $\exp(-\epsilon^2(k^2+p^2))$ can be introduced, and the limit $\epsilon\to 0$ taken outside the integral, since, by (A.1), the k- and p-integrals in (A.15) are absolutely convergent. By (A.2) and (A.3), the r- and s-integrals in (A.16) are absolutely convergent, so that, for $\epsilon > 0$, we can interchange the order of integration. The k- and p-integrals can be evaluated using (38), with the result

$$\begin{aligned}\langle\psi|\mathcal{F}|\psi\rangle &= \frac{2\hbar^2}{M}\sin\pi\tilde{\alpha}\lim_{\epsilon\to 0}\int_0^\infty\left(\frac{\psi_a(r)}{r^{\tilde{\alpha}}}\right)\mathrm{e}^{-r^2/4\epsilon^2}\left(\frac{r^2}{4\epsilon^2}\right)^{\tilde{\alpha}}\mathrm{d}\left(\frac{r^2}{4\epsilon^2}\right)\\ &\quad\times\int_0^\infty\left(\frac{\psi_{a+1}^*(s)}{s^{1-\tilde{\alpha}}}\right)\mathrm{e}^{-s^2/4\epsilon^2}\left(\frac{s^2}{4\epsilon^2}\right)^{1-\tilde{\alpha}}\mathrm{d}\left(\frac{s^2}{4\epsilon^2}\right)\\ &= \frac{2\pi\hbar^2}{M}\sin\pi\tilde{\alpha}C_aC_{a+1}^*\Gamma(1+\tilde{\alpha})\Gamma(2-\tilde{\alpha})\\ &= \frac{4\pi\hbar^2}{M}\tilde{\alpha}(1-\tilde{\alpha})C_aC_{a+1}^*\end{aligned} \tag{A.17}$$

where the coefficients C_a and C_{a+1} are given in (A.3). This is just the result (21) of section 2.

Concerning the impulse, it is straightforward to justify, using arguments like the preceding ones, the calculations of section 3 leading to (46). It is only necessary to check that the time-dependent expectation value, $\langle\psi(t)|\mathcal{F}|\psi(t)\rangle$, is integrable in t. $\langle\psi(t)|\mathcal{F}|\psi(t)\rangle$ is given by an expression like (A.15), but with $c_a(k)$ and $c_{a+1}^*(p)$ modulated by the factors $\exp(-\mathrm{i}\hbar k^2 t/2M)$ and $\exp(\mathrm{i}\hbar p^2 t/2M)$ respectively. We have that

$$\int_0^\infty k^{1+\tilde{\alpha}}c_m(k)\exp\left(-\mathrm{i}\frac{\hbar k^2}{2M}t\right)\mathrm{d}k = \frac{\mathrm{i}M}{\hbar t}\int_0^\infty\frac{\mathrm{d}}{\mathrm{d}k}(k^{\tilde{\alpha}}c_m(k))\exp\left(-\mathrm{i}\frac{\hbar k^2}{2M}t\right)\mathrm{d}k = \mathrm{O}(1/t) \tag{A.18}$$

for large t (the integration by parts is justified by (A.2) and (A.3)). Similarly, $\int_0^\infty p^{2-\tilde{\alpha}} c^*_{a+1}(p) \exp(\mathrm{i}\hbar p^2 t/2M)\,\mathrm{d}p = \mathrm{O}(1/t)$. It follows that $\langle\psi(t)|\mathcal{F}|\psi(t)\rangle$ falls off as $1/t^2$.

Appendix A.1. Proof of (A.13)

Given functions f and g defined on a domain D, we will say that f is dominated by g if, for some constant C, $|f| < Cg$ throughout D. For functions indexed by an integer m, e.g. f_m and g_m, we will say that f_m is dominated by g_m if $|f_m| < Cg_m$ for some C which does not depend on m. Thus, from (A.1),

$$c_m(k)c^*_{m+1}(p) \text{ is dominated by } (1+m^2)^{-1}(1+k^2+p^2)^{-4} \text{ for } k,\, p > 0. \tag{A.19}$$

The integral in (A.11) can be evaluated (Gradshteyn and Ryzhik 1980) to give

$$K^\epsilon_m(k,p) = \frac{\mathrm{sgn}(m-a)}{2\epsilon^2} k^2 p(k^2-p^2)\mathrm{e}^{-(k^2+p^2)/4\epsilon^2} I_\nu\left(\frac{kp}{2\epsilon^2}\right) \tag{A.20}$$

where $I_\nu(z)$ is a modified Bessel function, and $\nu = |m-\alpha|$. From the asymptotic behaviour of $I_\nu(z)$ for large argument, it follows that $I_\nu(z)$ is dominated by $\mathrm{e}^z/\sqrt{z}$ for z real. Therefore, the left-hand side of (A.13) is dominated by

$$\sum_{m=-\infty}^{\infty} \frac{1}{1+m^2}\frac{1}{\epsilon}\int_0^\infty\int_0^\infty \exp\left(-\frac{(k-p)^2}{2\epsilon^2}\right)\frac{k^{3/2}p^{1/2}|k^2-p^2|}{(1+k^2+p^2)^4}\,\mathrm{d}k\,\mathrm{d}p. \tag{A.21}$$

Let us divide the domain of the (k, p)-integral into regions inside and outside the strip $|k-p| < \epsilon^{2/3}$. Inside, the integrand is dominated by $\epsilon^{2/3}k^3/(1+k^2)^4$; thus the integral over the strip is dominated by $\epsilon^{4/3}$. Outside the strip, the integrand is dominated by $\exp(-\epsilon^{-2/3})/(1+k^2+p^2)^2$; thus the (k, p)-integral outside the strip is dominated by $\exp(-\epsilon^{-2/3})$. Therefore, the expression in (A.21) is dominated by

$$\sum_{m=-\infty}^{\infty} \frac{1}{1+m^2}\frac{1}{\epsilon}(\epsilon^{4/3} + \exp(-\epsilon^{-2/3})) \tag{A.22}$$

which vanishes as $\epsilon \to 0$.

Appendix A.2. Proof of (A.14)

Substituting the series expansion

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{u=0}^{\infty}\frac{1}{u!\Gamma(u+\nu+1)}\left(-\frac{z^2}{4}\right)^u \tag{A.23}$$

and a similar expansion for $J_{-\nu}(z)$, we obtain that

$$\begin{aligned} k^\nu &\int_0^\infty \mathrm{e}^{-\epsilon^2 r^2} J_\nu(pr)J_{-\nu}(kr)r\,\mathrm{d}r \\ &= \frac{1}{2\epsilon^2}p^\nu \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{\Gamma(u+v+1)}{u!v!\Gamma(u-\nu+1)\Gamma(v+\nu+1)}\left(-\frac{k^2}{4\epsilon^2}\right)^u\left(-\frac{p^2}{4\epsilon^2}\right)^v. \end{aligned} \tag{A.24}$$

For $\epsilon > 0$, the r-integral and u- and v-sums are absolutely convergent for all $k, p \geqslant 0$. Inserting in (A.24) the integral representation for the reciprocal of the beta-function (Gradshteyn and Ryzhik 1980)

$$\begin{aligned} \frac{\Gamma(u+v+1)}{\Gamma(u-\nu+1)\Gamma(v+\nu+1)} &= \frac{1}{(u+v+1)B(u-\nu+1, v+\nu+1)} \\ &= \frac{2}{\pi}\mathrm{Re}\int_0^{\pi/2}(2\mathrm{i}\sin\tau\,\mathrm{e}^{-\mathrm{i}\tau})^u(-2\mathrm{i}\sin\tau\,\mathrm{e}^{\mathrm{i}\tau})^v\mathrm{e}^{2\mathrm{i}\nu(\tau-\pi/2)}\,\mathrm{d}\tau \end{aligned} \tag{A.25}$$

we may perform the sums to obtain

$$k^{\nu}\int_0^{\infty} e^{-\epsilon^2 r^2} J_{\nu}(pr)J_{-\nu}(kr)r\,dr = \frac{p^{\nu}}{\pi\epsilon^2}\mathrm{Re}\int_0^{\pi/2}\exp\biggl(-\frac{p^2+k^2}{2\epsilon^2}\sin^2\tau + i\frac{p^2-k^2}{4\epsilon^2}\sin 2\tau + 2i\nu(\tau-\pi/2)\biggr)\,d\tau. \tag{A.26}$$

Substituting this result into (A.12), we obtain

$$K_a^{\epsilon}(k,p) = \frac{1}{\pi\epsilon^2}k^{1+\tilde{\alpha}}p^{2-\tilde{\alpha}}(p^2-k^2)\mathrm{Re}\int_0^{\pi/2} e^{-S/\epsilon^2}\,d\tau \tag{A.27}$$

where the exponent S is given by

$$S = \tfrac{1}{4}(k^2+p^2)(1-\cos 2\tau) - \tfrac{1}{4}i(p^2-k^2)\sin 2\tau - 2i\epsilon^2(1-\tilde{\alpha})(\tau-\pi/2). \tag{A.28}$$

It is clear that the main contribution to the τ-integral in (A.27) comes from the neighbourhood of $\tau = 0$. If S is expanded about $\tau = 0$ to second order, the τ-integral yields an error function, whose leading-order asymptotics as $\epsilon \to 0$ leads directly to the required result (A.14). However, the next term in the asymptotic expansion is not uniformly bounded in k and p—it contains a factor $(k^2-p^2)^{-2}$—so we must take some additional care.

To proceed, we divide the domain of the k, p-integral into the three regions specified below, writing the left-hand side of (A.14) as

$$\biggl(\lim_{\epsilon\to 0}\iint_{D_1} + \lim_{\epsilon\to 0}\iint_{D_2} + \lim_{\epsilon\to 0}\iint_{D_3}\biggr)K_a^{\epsilon}(k,p)c_{a+1}^{*}(p)c_a(k)\,dk\,dp \tag{A.29}$$

and analysing the contribution from each region separately.

Let D_1 denote the region $k, p \geqslant 0$, $k^2+p^2 \leqslant \epsilon^{\beta}$, where β is chosen to satisfy

$$\tfrac{4}{7} < \beta < \tfrac{2}{3}. \tag{A.30}$$

In this region, the exponential factor in (A.27) is bounded, and $k^{1+\tilde{\alpha}}p^{2-\tilde{\alpha}}(p^2-k^2)$ is dominated by $\epsilon^{5\beta/2}$. The coefficients $|c_a(k)|$ and $|c_{a+1}^{*}(p)|$ are bounded, and the area of D_1 is dominated by $\epsilon^{2\beta}$, so that the contribution from D_1 in (A.29) is dominated by $\epsilon^{7\beta/2-2}$. Given the choice of β, this vanishes as $\epsilon \to 0$.

Let D_2 be the region $k, p \geqslant 0$, $k^2+p^2 \geqslant \epsilon^{\beta}$ and $|p^2-k^2| \leqslant \epsilon^{\gamma}$, where γ is chosen to satisfy

$$\tfrac{1}{2} + \tfrac{1}{4}\beta < \gamma < 1 - \tfrac{1}{2}\beta \tag{A.31}$$

(since $\beta < \frac{2}{3}$, the inequality (A.31) can be always be satisfied). Since $1-\cos 2\tau \geqslant \tau^2$ for $0 \leqslant \tau \leqslant \pi/2$, the factor $\exp(-S/\epsilon^2)$ is dominated by the Gaussian $\exp(-\sigma^2\tau^2)$, where $\sigma = \frac{1}{2}\epsilon^{\beta/2-1}$. Thus $\int_0^{\pi/2}\exp(-S/\epsilon^2)\,d\tau$ is dominated by $\epsilon^{1-\beta/2}$. Then, from (A.27), $K_a^{\epsilon}(k,p)$ is dominated by $\epsilon^{\gamma-\beta/2-1}k^{1+\alpha}p^{2-\alpha}$ in D_2. From (A.1), $p^{2-\alpha}c_{a+1}^{*}(p)c_a(k)$ is integrable over the region $k, p \geqslant 0$. Therefore, the contribution from D_2 to (A.29) is dominated by $\epsilon^{2\gamma-\frac{1}{2}\beta-1}$ (the additional factor of ϵ^{γ} is due to the fact that the integral is confined to $|k^2-p^2| \leqslant \epsilon^{\gamma}$). Given the choice of γ, this vanishes as $\epsilon \to 0$.

The remaining region D_3 is given by $k, p \geqslant 0$, $k^2+p^2 \geqslant \epsilon^{\beta}$ and $|p^2-k^2| \geqslant \epsilon^{\gamma}$. Integrating by parts with respect to τ in (A.27), we obtain that

$$K_a^{\epsilon}(k,p) = \frac{1}{\pi}k^{1+\tilde{\alpha}}p^{2-\tilde{\alpha}}(p^2-k^2) \times\mathrm{Re}\biggl(\frac{e^{-S/\epsilon^2}}{S_{\tau}}\bigg|_{\tau=0} - \frac{e^{-S/\epsilon^2}}{S_{\tau}}\bigg|_{\tau=\pi/2} - \int_0^{\pi/2} e^{-S}\frac{S_{\tau\tau}}{S_{\tau}^2}\,d\tau\biggr). \tag{A.32}$$

The first term gives

$$\frac{2}{\pi}\sin\pi\tilde{\alpha}\,k^{1+\tilde{\alpha}}p^{2-\tilde{\alpha}}\left(1+\frac{4(1-\tilde{\alpha})\epsilon^2}{p^2-k^2}\right)^{-1}. \tag{A.33}$$

Its contribution to the integral over D_3 in (A.29) yields, in the limit $\epsilon \to 0$, the required result (A.14).

It remains to show that the contribution to the D_3-integral from the remaining terms in (A.32) vanishes in the limit $\epsilon \to 0$. It is readily seen that the contribution from the second term vanishes exponentially with ϵ. For the third term, we note that, on the interval $0 \leqslant \tau \leqslant \pi/2$, $S_{\tau\tau}/S_\tau^2$ is dominated by $(k^2+p^2)/(k^2-p^2)^2$, and $\exp(-S/\epsilon^2)$ is dominated by $\exp(-\epsilon^{\beta-2}\tau^2/2)$. Therefore, the integral $\int_0^{\pi/2} \mathrm{e}^{-S/\epsilon^2}(S_{\tau\tau}/S_\tau^2)\,\mathrm{d}\tau$ is dominated by $\epsilon^{1-\beta/2}(k^2+p^2)/(p^2-k^2)^2$. Thus, the third term in (A.32) is dominated by $\epsilon^{1-\beta/2}(k^2+p^2)k^{1+\tilde{\alpha}}p^{2-\tilde{\alpha}}/|k^2-p^2|$, which on D_3 is dominated by $\epsilon^{1-\beta/2-\gamma}(k^2+p^2)^{7/2}$. The contribution to the integral over D_3 in (A.29) is dominated by $\epsilon^{1-\beta/2-\gamma}$, which, by the choice of γ, vanishes as $\epsilon \to 0$.

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